Math 254B Lecture 25 Notes

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May 29, 2019

1 Dimension of Attractor Systems Over a Factor Map

1.1 Dimension of attractor systems over a factor map

Let $\Phi = (\Phi_i)_{i=1}^k$ be an IFS with attractor K. Let $\pi : [k]^{\mathbb{N}} \to K$ be the coding map, which sends $\pi(\omega) = \lim_n \Phi_{\omega|_1^n}(0)$. We are making the following assumptions for the remainder of the course:

1. Same ratio: $\Phi_i x = rU_i x + a_i$
2. SSC: $\implies$ π is a conjugacy $([k]^{\mathbb{N}}, \sigma) \to (K, S)$ where $S|_{K_i} = (\Phi_i|_K)^{-1}$.

We had the following theorems.

Theorem 1.1.

$$\dim(K) = \frac{h_{\mathrm{top}}(K, S)}{\log(r^{-1})} = \frac{\log(k)}{\log(r^{-1})}.$$

If $\mu \in P_e^S(K)$, then

$$\overline{\dim}(\mu) = \underline{\dim}(\mu) = \frac{h(\mu, S)}{\log(r^{-1})}.$$

We will need the following generalization.

Theorem 1.2. *Assume $\mu \in P_e^S(K)$ and $(K, \mu, S) \xrightarrow{\beta = \langle \beta_1, \beta_2, \dots \rangle} (\Sigma_\ell, \nu, \sigma)$ is a factor map. Then, if we disintegrate $\mu = \int_{\Sigma_\ell} \mu_y \, d\nu(y)$, then*

$$\dim(\mu_y) = \frac{\dim(\mu, S \mid \beta)}{\log(r^{-1})} \qquad \textit{for } \nu\textit{-a.e. } y.$$

Here, $h(\mu, S \mid \beta) = h(\mu, S) - h(\mu, \beta) = h(\mu, S) - h(\nu, \sigma)$.

Proof. We will prove that $\overline{\dim}(\mu) \leq \frac{\dim(\mu,S|\beta)}{\log(r^{-1})}$ a.s. We find $Y \subseteq \sum_\ell$ with $\nu(Y) = 1$ such that for all $y \in Y$, there is a $W_y \subseteq K$ with $\underline{\dim}_B(W_y) \leq h/\log(r)^{-1}$ and $\mu_y(W_y) = 1$.

Step 1: The inverse of the coding map $K \xrightarrow{\alpha=\langle\alpha_1,\alpha_2,\ldots\rangle} [k]^{\mathbb{N}}$ is $\alpha_1(i) = i)$ iff $z \in K_i = \Phi_i[k]$. Then $\alpha_2(z) = i$ iff $\alpha_1(Sz) = i$ iff $Sz \in K_i$ iff $Z \in K_{j,i}$ for some $j \in [k]$. So $\alpha = \pi^{-1}$. This gives $h = h(\alpha,\mu,S) - h(\beta,\mu,S) = h((\alpha,\beta),\mu,S) - h(\beta,\mu,S)$.

Notation: set $K_a = \{z \in L : \alpha_{[1;n]}(z) = a\} = [a]$ and $\{z \in K : \beta_{[1;n]}(z) = b\} = [b]$.

Step 2:

$$\mu([\alpha_{1,n}(z)] \mid [\beta_{1,n}(z)]) = \frac{\mu([\alpha_{1,n}(z)] \cap [\beta_{1,n}(z)])}{\mu([\beta_{1,n}(z)])}$$

Apply the Shannon-McMillan-Beriman theorem upstairs and downstairs.

$$\begin{aligned} &= \frac{e^{-h((\alpha,\beta),\mu,S)n+o(n)}}{e^{-h(\beta,\mu,S)n+o(n)}} \\ &= e^{-hn+oz(n)} \end{aligned}$$

for a.e. z. If $b \in [\ell]^n$, let $G_b = \{a : \mu([a] \mid [b]) > e^{-hn-\varepsilon n}\}$. So a.e. z is such that $\alpha_{[1,n]}(z) \in G_{\beta_{[1,n]}(z)}$ for all sufficiently large n. and $|G_b| < e^{hn+\varepsilon n}$. Let $W_b = \{z \in K : \alpha_{[1,n]}(z) \in G_b\}$. So $\mu(\{z : z \in W_{\beta_{[1,n]}(z)}\}) \xrightarrow{n\to\infty} 1$.

Step 3: Fix $b \in [\ell]^n$, and consider the set $W_b = \bigcup_{a \in G_b} K_a = \bigcup_{a\in G_b} \Phi_a[K]$. The sets in this union have diameter Cr^n. So we get

$$\operatorname{cov}_{Cr^n}(W_b) \leq |G_b| < e^{hn+o(n)}.$$

Step 4: $\mu = \int_{\Sigma_\ell} \mu_y \, d\nu(y)$, so

$$\mu(\{z \in W_{\beta_{1,n}(z)}\}) = \int_{\Sigma_\ell} \mu_y(\{z \in W_{(y_1,\ldots,y_n)}\}) \, d\nu(y) \to 1.$$

So there exist $n_1 < n_2 < \cdots$ such that for all i,

$$\nu(\{y : \mu_y(W_{(y_1,\ldots,y_n)}) > 1 - 2^{-i}\}) > 1 - 2^{-i}.$$

By Borel-Cantelli, $\nu(Y) = 1$, where $Y = \{y : \mu_y(W_{(y_1,\ldots,y_n)}) > 1 - 2^{-i}$ for all large $i\}$. But if $y \in Y$ and this condition holds for all $i \geq j$, then

$$\underbrace{\mu\left(\bigcup_{j\geq 1} \bigcap_{i \geq j} W_{(y_1,\ldots,y_n)}\right)}_{W_y} > 1 - 2^{-j} - 2^{-j-1} - \cdots = 1 - 2^{-j+1}.$$

So

$$\dim_H(W_j) = \sup_j \dim\left(\bigcap_{i\geq j} W_{(y_1,\ldots,y_n)}\right).$$

Finally, for all $i \geq j$, we have

$$\mathrm{cov}_{Cr^{n_i}}\left(\bigcap_{i\geq j} W_{(y_1,\ldots,y_n)}\right) \leq \mathrm{cov}_{Cr^{n_i}}(W_{(y_1,\ldots,y_{n_i})}) < e^{(h+\varepsilon)n_i} = e^{-n(h+\varepsilon)/\log(r^{-1})}.$$

So we get

$$\underline{\dim}_B\left(\bigcap_{i\geq j} W_{(y_1,\ldots,y_n)}\right) \leq \mathrm{cov}_{Cr^{n_i}}(W_{(y_1,\ldots,y_{n_i})}) < e^{(h+\varepsilon)n_i} \leq \frac{h+\varepsilon}{\log(r^{-1})}$$

□

1.2 The destination: theorems from Wu's paper

We will now restrict to $\mathbb{R}^2$ and make a third assumption: that if $\Phi_i x = rU_i x + a_i$., then all the U_i are the same rotation by $2\pi\xi$, where $\xi \in \mathbb{R} \setminus \mathbb{Q}$.

Theorem 1.3. *Under our 3 assumptions, we have*

$$\dim(K \cap L) \leq \max\{0, \dim(K) - 1\}$$

for every line $L \subseteq \mathbb{R}^2$.

We will prove this theorem because it has the idea of the proof we want, but the actual big theorem from Wu's paper is the following:

Theorem 1.4. *Let $p, q \in \mathbb{N}$ be coprime ($\log(p)/\log(q) \notin \mathbb{Q}$). Let $A, B \in \mathbb{T}$ be closed, and let $T_p[A] = A$ and $T_q[B] = B$. Then*

$$\dim((A \times B) \cap L) \leq \max\{0, \dim(A) + \dim(B) - 1\}$$

for all L not parallel to the x-axis or the y-axis.

Why is this a natural thing to think about? If $L = \{(x, ax+b) : x\ in \mathbb{R}\}$ with $a \neq 0$¡ then the projection of $(A \times B \cap L$ to the y-axis is $\{y : (a^{-1}(y-b), y) \in A \times B\} = (aA+b) \cap B$.